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Intuitionistic fuzzy ideals in unital cycloids

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Abstract. The notion of intuitionistic fuzzy ideal in unital cycloids based on the fuzzy points is introduced, the related properties are investigated. Characterizations of intuitionistic fuzzy ideal are given, conditions for an intuitionistic fuzzy set to be intuitionistic fuzzy ideal are provided. The $\in_{(t,s)}$-level set is defined, condition in which it becomes ideal is discussed.

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1. Introduction

In the study of integrable systems, topological quantum fields and quantum groups, the quantum Yang-Baxter equation plays an important role. The solving of the quantum Yang-Baxter equations is a main topic in the mathematical physics, because it is closely related to algebra, topology and physics. In [1, 2], the algebraic solution of the quantum Yang-Baxter equation is studied by W. Rump, using the notion of unital cycloids and L-algebras. The L-algebras are consistently studied in the field of logical algebras (See [3, 4]). Also, the study of (fuzzy) ideals in unital cycloids is introduced by Y. B. Jun in [5]. Meanwhile, research related to fuzzy theory has been steadily conducted until recently (See [6, 7, 8]). In this paper, I study intuitionistic fuzzy ideals in unital cycloids, by using the fuzzy set theory based on the fuzzy point. I introduce the notion of intuitionistic fuzzy ideal in unital cycloid, and investigate several properties. I give characterizations of intuitionistic fuzzy ideals, and provide the conditions under which an intuitionistic fuzzy set can be an intuitionistic fuzzy ideal. I consider the $\in_{(t,s)}$-level set, and provide condition in which it becomes ideal.

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2. Preliminaries

2.1. **Basic results on L-algebras.** What is already known about L-algebras is mentioned here. For more information, please refer to [2, 4, 9], etc.

Definition 2.1 ([2, 9])**.** Let L be a set with a binary operation " $\rightarrow$ ". An element $\ell \in L$ is called a *logical unit*, if it satisfies:

$$(\forall \mathfrak{a} \in L)(\ell \rightarrow \mathfrak{a} = \mathfrak{a},\ \mathfrak{a} \rightarrow \mathfrak{a} = \mathfrak{a} \rightarrow \ell = \ell). \tag{2.1}$$

A couple $(L, \rightarrow)$ is called a *cycloid*, if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)((\mathfrak{a} \rightarrow \mathfrak{b}) \rightarrow (\mathfrak{a} \rightarrow \mathfrak{c}) = (\mathfrak{b} \rightarrow \mathfrak{a}) \rightarrow (\mathfrak{b} \rightarrow \mathfrak{c})). \tag{2.2}$$

If a cycloid $(L, \rightarrow)$ has a logical unit ℓ, then it is called a *unital cycloid* and is denoted by $\mathcal{L} :=(L, \rightarrow, \ell)$.

Definition 2.2 ([2, 9])**.** An *L-algebra* is defined to be a unital cycloid $\mathcal{L} :=(L, \rightarrow, \ell)$ that satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \rightarrow \mathfrak{b} = \mathfrak{b} \rightarrow \mathfrak{a} = \ell \ \Rightarrow \ \mathfrak{a} = \mathfrak{b}). \tag{2.3}$$

We define a binary relation " $\leq_L$ " in an L-algebra $\mathcal{L} :=(L, \rightarrow, \ell)$ as follows.

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \leq_L \mathfrak{b} \ \Leftrightarrow \ \mathfrak{a} \rightarrow \mathfrak{b} = \ell). \tag{2.4}$$

If $\mathcal{L} :=(L, \rightarrow, \ell)$ is an L-algebra, then $(L, \leq_L)$ is a poset.

Definition 2.3 ([2])**.** Let $\mathcal{L} :=(L, \rightarrow, \ell)$ be a unital cycloid. A subset I of L is called an *ideal* of $\mathcal{L}$, if it satisfies:

$$\ell \in I, \tag{2.5}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \in I,\ \mathfrak{a} \rightarrow \mathfrak{b} \in I \ \Rightarrow \ \mathfrak{b} \in I), \tag{2.6}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \in I \ \Rightarrow \ (\mathfrak{a} \rightarrow \mathfrak{b}) \rightarrow \mathfrak{b} \in I), \tag{2.7}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \in I \ \Rightarrow \ \mathfrak{b} \rightarrow \mathfrak{a} \in I,\ \mathfrak{b} \rightarrow (\mathfrak{a} \rightarrow \mathfrak{b}) \in I). \tag{2.8}$$

2.2. **Basic results on the (intuitionistic) fuzzy set theory.** A function $f : L \rightarrow [0, 1]$ is called a *fuzzy set* in a set L, and the *complement* of f is denoted by $\neg f$, and is given as follows:

$$\neg f : L \rightarrow [0, 1],\ x \mapsto 1 - f(x).$$

For every fuzzy sets f and g in L, we say $f \leq g$ if $f(x) \leq g(x)$ for all $x \in L$.

A fuzzy set f in a set L of the form

$$f(a) := \begin{cases} t \in (0, 1] & \text{if } a = c \\ 0 & \text{if } a \neq c \end{cases}$$

is said to be a *fuzzy point* with support c and value t and is denoted by c_t.

The concept of intuitionistic fuzzy set was introduced by Atanassov (See [10, 11, 12]) as follows: An *intuitionistic fuzzy set* (briefly, *IF-set*) on a set L is an expression $\mathbf{I}$ given by

$$\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$$

where $f_{\mathbf{I}}$ and $g_{\mathbf{I}}$ are fuzzy sets in L such that $0 \leq f_{\mathbf{I}}(x) + g_{\mathbf{I}}(x) \leq 1$ for all $x \in L$.

Every fuzzy set f in a set L is obviously an intuitionistic fuzzy set having the form $\{\langle x, f, \neg f\rangle \mid x \in L\}$ (See [11]).

The notion of intuitionistic fuzzy point is considered in the paper [13, 14] as follows: Given elements $a \in L$ and $(t, s) \in (0, 1] \times [0, 1)$ satisfying $t + s \leq 1$, the intuitionistic fuzzy set

$$a_{(t,s)} := \{\langle x, a_t, \neg a_{1-s}\rangle \mid x \in L\} \tag{2.9}$$

is called an *intuitionistic fuzzy point* in L.

Let $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ be an IF-set in L. An intuitionistic fuzzy point $a_{(t,s)} := \{\langle x, a_t, \neg a_{1-s}\rangle \mid x \in L\}$ is said to be

- *contained* in $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$, denoted by $a_{(t,s)} \in \mathbf{I}$, if $a_t \leq f_{\mathbf{I}}$ and $\neg a_{1-s} \geq g_{\mathbf{I}}$, or equivalently, $f_{\mathbf{I}}(a) \geq t$ and $g_{\mathbf{I}}(a) \leq s$.
- *quasi-coincident* with $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$, denoted by $a_{(t,s)}\, q\, \mathbf{I}$, if $f_{\mathbf{I}}(a) + t > 1$ and $g_{\mathbf{I}}(a) + s < 1$.

If $a_{(t,s)}\, \beta\, \mathbf{I}$ is not established for $\beta \in \{\in, q\}$, it is denoted by $a_{(t,s)}\, \overline{\beta}\, \mathbf{I}$. The set

$$\mathbf{I}^{\in}_{(t,s)} := \{a \in L \mid a_{(t,s)} \in \mathbf{I}\}$$

is called the $\in_{(t,s)}$*-level set* of $\mathbf{I}$. It is clear that

$$\mathbf{I}^{\in}_{(t,s)} = U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$$

where $U(f_{\mathbf{I}}, t) := \{c \in L \mid f_{\mathbf{I}}(c) \geq t\}$ and $L(g_{\mathbf{I}}, s) := \{c \in L \mid g_{\mathbf{I}}(c) \leq s\}$, which are called the *upper t-level set* and the *lower s-level set* of $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$.

3. Intuitionistic fuzzy ideals

We define intuitionistic fuzzy ideal in a unital cycloid based on intuitionistic fuzzy points.

Definition 3.1. Let $\mathcal{L} := (L, \rightarrow, \ell)$ be a unital cycloid. An IF-set $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ in L is called an *intuitionistic fuzzy ideal* (briefly, *IF-ideal*) of $\mathcal{L} := (L, \rightarrow, \ell)$, if

$$x_{(t,s)} \in \mathbf{I} \;\Rightarrow\; \ell_{(t,s)} \in \mathbf{I}, \tag{3.1}$$

$$x_{(t_1,s_1)} \in \mathbf{I},\ (x \rightarrow y)_{(t_2,s_2)} \in \mathbf{I} \;\Rightarrow\; y_{(\min\{t_1,t_2\},\max\{s_1,s_2\})} \in \mathbf{I}, \tag{3.2}$$

$$x_{(t,s)} \in \mathbf{I} \;\Rightarrow\; ((x \rightarrow y) \rightarrow y)_{(t,s)} \in \mathbf{I}, \tag{3.3}$$

$$x_{(t,s)} \in \mathbf{I} \;\Rightarrow \begin{pmatrix} (y \rightarrow x)_{(t,s)} \in \mathbf{I}, \\ (y \rightarrow (x \rightarrow y))_{(t,s)} \in \mathbf{I} \end{pmatrix} \tag{3.4}$$

for all $x, y \in L$ and $(t, s), (t_i, s_i) \in (0, 1] \times [0, 1)$ where $i = 1, 2$.

We first present the necessary and sufficient conditions for the IF-set to be an IF-ideal.

Theorem 3.2. *Let $\mathcal{L}$:=$(L, \rightarrow, \ell)$ be a unital cycloid. An IF-set $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ in L is an IF-ideal of $\mathcal{L}$:=$(L, \rightarrow, \ell)$ if and only if it satisfies:*

$$(\forall x \in L)(f_{\mathbf{I}}(\ell) \geq f_{\mathbf{I}}(x),\ g_{\mathbf{I}}(\ell) \leq g_{\mathbf{I}}(x)), \tag{3.5}$$

$$(\forall x, y \in L)\begin{pmatrix} f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\} \\ g_{\mathbf{I}}(y) \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \rightarrow y)\} \end{pmatrix}, \tag{3.6}$$

$$(\forall x, y \in L)(f_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \geq f_{\mathbf{I}}(x),\ g_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \leq g_{\mathbf{I}}(x)), \tag{3.7}$$

$$(\forall x, y \in L)\begin{pmatrix} f_{\mathbf{I}}(y \rightarrow x) \geq f_{\mathbf{I}}(x),\ g_{\mathbf{I}}(y \rightarrow x) \leq g_{\mathbf{I}}(x) \\ f_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \geq f_{\mathbf{I}}(x),\ g_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \leq g_{\mathbf{I}}(x) \end{pmatrix}. \tag{3.8}$$

Proof. Assume that $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ in L is an IF-ideal of $\mathcal{L}$:=$(L, \rightarrow, \ell)$. Since $x_{(f_{\mathbf{I}}(x), g_{\mathbf{I}}(x))} \in \mathbf{I}$ for all $x \in L$, it follows from (3.1), (3.3) and (3.4) that $\ell_{(f_{\mathbf{I}}(x), g_{\mathbf{I}}(x))} \in \mathbf{I}$, $((x \rightarrow y) \rightarrow y)_{(f_{\mathbf{I}}(x), g_{\mathbf{I}}(x))} \in \mathbf{I}$, $(y \rightarrow x)_{(f_{\mathbf{I}}(x), g_{\mathbf{I}}(x))} \in \mathbf{I}$ and $(y \rightarrow (x \rightarrow y))_{(f_{\mathbf{I}}(x), g_{\mathbf{I}}(x))} \in \mathbf{I}$. Then

$$\begin{aligned} &f_{\mathbf{I}}(\ell) \geq f_{\mathbf{I}}(x),\ g_{\mathbf{I}}(\ell) \leq g_{\mathbf{I}}(x), \\ &f_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \geq f_{\mathbf{I}}(x),\ g_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \leq g_{\mathbf{I}}(x), \\ &f_{\mathbf{I}}(y \rightarrow x) \geq f_{\mathbf{I}}(x),\ g_{\mathbf{I}}(y \rightarrow x) \leq g_{\mathbf{I}}(x), \\ &f_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \geq f_{\mathbf{I}}(x),\ g_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \leq g_{\mathbf{I}}(x) \end{aligned}$$

for all $x, y \in L$. Note that $x_{(f_{\mathbf{I}}(x), g_{\mathbf{I}}(x))} \in \mathbf{I}$ and $(x \rightarrow y)_{(f_{\mathbf{I}}(x \rightarrow y), g_{\mathbf{I}}(x \rightarrow y))} \in \mathbf{I}$ for all $x, y \in L$. Thus $y_{(\min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\}, \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \rightarrow y)\})} \in \mathbf{I}$ by (3.2). So $f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\}$ and $g_{\mathbf{I}}(y) \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \rightarrow y)\}$.

Conversely, let $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ be an IF-set in L that satisfies (3.5), (3.6), (3.7), and (3.8). Let $x \in L$ and $(t, s) \in (0, 1] \times [0, 1)$ be such that $x_{(t,s)} \in \mathbf{I}$. Then $f_{\mathbf{I}}(x) \geq t$ and $g_{\mathbf{I}}(x) \leq s$. If we use (3.5), (3.7) and (3.8), it leads to

$$\begin{aligned} &f_{\mathbf{I}}(\ell) \geq f_{\mathbf{I}}(x) \geq t,\ g_{\mathbf{I}}(\ell) \leq g_{\mathbf{I}}(x) \leq s, \\ &f_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \geq f_{\mathbf{I}}(x) \geq t,\ g_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \leq g_{\mathbf{I}}(x) \leq s, \\ &f_{\mathbf{I}}(y \rightarrow x) \geq f_{\mathbf{I}}(x) \geq t,\ g_{\mathbf{I}}(y \rightarrow x) \leq g_{\mathbf{I}}(x) \leq s, \\ &f_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \geq f_{\mathbf{I}}(x) \geq t,\ g_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \leq g_{\mathbf{I}}(x) \leq s \end{aligned}$$

for all $y \in L$. Thus $\ell_{(t,s)} \in \mathbf{I}$, $((x \rightarrow y) \rightarrow y)_{(t,s)} \in \mathbf{I}$, $(y \rightarrow x)_{(t,s)} \in \mathbf{I}$ and $(y \rightarrow (x \rightarrow y))_{(t,s)} \in \mathbf{I}$. Now, let $x, y \in L$ and $(t_1, s_1), (t_2, s_2) \in (0, 1] \times [0, 1)$ be such that $x_{(t_1,s_1)} \in \mathbf{I}$ and $(x \rightarrow y)_{(t_2,s_2)} \in \mathbf{I}$. Then $f_{\mathbf{I}}(x) \geq t_1$, $g_{\mathbf{I}}(x) \leq s_1$, $f_{\mathbf{I}}(x \rightarrow y) \geq t_2$, and $g_{\mathbf{I}}(x \rightarrow y) \leq s_2$. It follows from (3.6) that

$$f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\} \geq \min\{t_1, t_2\}$$

and

$$g_{\mathbf{I}}(y) \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \rightarrow y)\} \leq \max\{s_1, s_2\}.$$

Thus $y_{(\min\{t_1,t_2\}, \max\{s_1,s_2\})} \in \mathbf{I}$. So $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ is an IF-ideal of $\mathcal{L}$:=$(L, \rightarrow, \ell)$. □

Theorem 3.3. *Let $\mathcal{L}$:=$(L, \rightarrow, \ell)$ be a unital cycloid. An IF-set $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ in L is an IF-ideal of $\mathcal{L}$:=$(L, \rightarrow, \ell)$ if and only if the nonempty upper t-level* *set* $U(f_{\mathbf{I}}, t)$ *and the nonempty lower* s*-level set* $L(g_{\mathbf{I}}, s)$ *are ideals of* $\mathcal{L} :=(L, \rightarrow, \ell)$ *for all* $(t, s) \in (0, 1] \times [0, 1)$ *satisfying* $t + s \leq 1$*.*

Proof. Assume that $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ is an IF-ideal of $\mathcal{L} :=(L, \rightarrow, \ell)$. Let $(t, s) \in (0, 1] \times [0, 1)$ be such that $t + s \leq 1$ and $U(f_{\mathbf{I}}, t) \neq \varnothing \neq L(g_{\mathbf{I}}, s)$. It is clear that $\ell \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$. Let $x, \mathfrak{a}, y, \mathfrak{b} \in L$. If $(x, \mathfrak{a}) \in U(f_{\mathbf{I}}, t) \times L(g_{\mathbf{I}}, s)$ and $(x \rightarrow y, \mathfrak{a} \rightarrow \mathfrak{b}) \in U(f_{\mathbf{I}}, t) \times L(g_{\mathbf{I}}, s)$, then $f_{\mathbf{I}}(x) \geq t$, $f_{\mathbf{I}}(x \rightarrow y) \geq t$, $g_{\mathbf{I}}(\mathfrak{a}) \leq s$, and $g_{\mathbf{I}}(\mathfrak{a} \rightarrow \mathfrak{b}) \leq s$. It follows from (3.6) that $f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\} \geq t$ and $g_{\mathbf{I}}(\mathfrak{b}) \leq \max\{g_{\mathbf{I}}(\mathfrak{a}), g_{\mathbf{I}}(\mathfrak{a} \rightarrow \mathfrak{b})\} \leq s$. Thus $(y, \mathfrak{b}) \in U(f_{\mathbf{I}}, t) \times L(g_{\mathbf{I}}, s)$. Let $(x, \mathfrak{a}) \in U(f_{\mathbf{I}}, t) \times L(g_{\mathbf{I}}, s)$. Then $f_{\mathbf{I}}(x) \geq t$ and $g_{\mathbf{I}}(\mathfrak{a}) \leq s$. Using (3.7) and (3.8), we have $f_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \geq f_{\mathbf{I}}(x) \geq t$, $g_{\mathbf{I}}((\mathfrak{a} \rightarrow \mathfrak{b}) \rightarrow \mathfrak{b}) \leq g_{\mathbf{I}}(\mathfrak{a}) \leq s$, $f_{\mathbf{I}}(y \rightarrow x) \geq f_{\mathbf{I}}(x) \geq t$, $g_{\mathbf{I}}(\mathfrak{b} \rightarrow \mathfrak{a}) \leq g_{\mathbf{I}}(\mathfrak{a}) \leq s$, $f_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \geq f_{\mathbf{I}}(x) \geq t$ and $g_{\mathbf{I}}(\mathfrak{b} \rightarrow (\mathfrak{a} \rightarrow \mathfrak{b})) \leq g_{\mathbf{I}}(\mathfrak{a}) \leq s$. Thus

$$((x \rightarrow y) \rightarrow y, (\mathfrak{a} \rightarrow \mathfrak{b}) \rightarrow \mathfrak{b}) \in U(f_{\mathbf{I}}, t) \times L(g_{\mathbf{I}}, s),$$

$(y \rightarrow x, \mathfrak{b} \rightarrow \mathfrak{a}) \in U(f_{\mathbf{I}}, t) \times L(g_{\mathbf{I}}, s)$ and

$$(y \rightarrow (x \rightarrow y), \mathfrak{b} \rightarrow (\mathfrak{a} \rightarrow \mathfrak{b})) \in U(f_{\mathbf{I}}, t) \times L(g_{\mathbf{I}}, s).$$

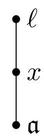
So $U(f_{\mathbf{I}}, t)$ and $L(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \rightarrow, \ell)$.

Conversely, suppose the nonempty upper t-level set $U(f_{\mathbf{I}}, t)$ and the nonempty lower s-level set $L(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \rightarrow, \ell)$ for all $(t, s) \in (0, 1] \times [0, 1)$ satisfying $t + s \leq 1$. For any $x \in L$, let $(f_{\mathbf{I}}(x), g_{\mathbf{I}}(x)) = (t, s)$. Then $x \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$, so $U(f_{\mathbf{I}}, t) \neq \varnothing \neq L(g_{\mathbf{I}}, s)$. Since $U(f_{\mathbf{I}}, t)$ and $L(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \rightarrow, \ell)$, we get $\ell \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$. Hence $f_{\mathbf{I}}(\ell) \geq t = f_{\mathbf{I}}(x)$ and $g_{\mathbf{I}}(\ell) \leq s = g_{\mathbf{I}}(x)$, so (3.5) is valid. If (3.6) is not valid, then there exist $x, y, \mathfrak{a}, \mathfrak{b} \in L$ such that $f_{\mathbf{I}}(y) < \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\}$ or $g_{\mathbf{I}}(\mathfrak{b}) > \max\{g_{\mathbf{I}}(\mathfrak{a}), g_{\mathbf{I}}(\mathfrak{a} \rightarrow \mathfrak{b})\}$. If we take $t_0 = \frac{1}{2}(f_{\mathbf{I}}(y) + \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\})$ or $s_0 = \frac{1}{2}(g_{\mathbf{I}}(\mathfrak{b}) + \max\{g_{\mathbf{I}}(\mathfrak{a}), g_{\mathbf{I}}(\mathfrak{a} \rightarrow \mathfrak{b})\})$, then $f_{\mathbf{I}}(y) < t_0 < \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\}$ or $g_{\mathbf{I}}(\mathfrak{b}) > s_0 > \max\{g_{\mathbf{I}}(\mathfrak{a}), g_{\mathbf{I}}(\mathfrak{a} \rightarrow \mathfrak{b})\}$. If $f_{\mathbf{I}}(y) < t_0 < \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \rightarrow y)\}$, then $x \in U(f_{\mathbf{I}}, t_0)$ and $x \rightarrow y \in U(f_{\mathbf{I}}, t_0)$, but $y \notin U(f_{\mathbf{I}}, t_0)$. If $g_{\mathbf{I}}(\mathfrak{b}) > s_0 > \max\{g_{\mathbf{I}}(\mathfrak{a}), g_{\mathbf{I}}(\mathfrak{a} \rightarrow \mathfrak{b})\}$, then $\mathfrak{a} \in L(g_{\mathbf{I}}, s_0)$ and $\mathfrak{a} \rightarrow \mathfrak{b} \in L(g_{\mathbf{I}}, s_0)$, but $\mathfrak{b} \notin L(g_{\mathbf{I}}, s_0)$. This is a contradiction to (2.6), so (3.6) is valid. If $f_{\mathbf{I}}((x \rightarrow y) \rightarrow y) < f_{\mathbf{I}}(x)$ or $g_{\mathbf{I}}((x \rightarrow y) \rightarrow y) > g_{\mathbf{I}}(x)$ for some $x, y \in L$, then $(x \rightarrow y) \rightarrow y \notin U(f_{\mathbf{I}}, f_{\mathbf{I}}(x))$ or $(x \rightarrow y) \rightarrow y \notin L(g_{\mathbf{I}}, g_{\mathbf{I}}(x))$. But $x \in U(f_{\mathbf{I}}, f_{\mathbf{I}}(x))$ and $x \in L(g_{\mathbf{I}}, g_{\mathbf{I}}(x))$. This is a contradiction to (2.7), so $f_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \geq f_{\mathbf{I}}(x)$ and $g_{\mathbf{I}}((x \rightarrow y) \rightarrow y) \leq g_{\mathbf{I}}(x)$ for all $x, y \in L$. Thus (3.7) is valid. If $f_{\mathbf{I}}(y \rightarrow x) < f_{\mathbf{I}}(x)$ or $g_{\mathbf{I}}(y \rightarrow x) > g_{\mathbf{I}}(x)$ for some $x, y \in L$, then $y \rightarrow x \notin U(f_{\mathbf{I}}, f_{\mathbf{I}}(x))$ or $y \rightarrow x \notin L(g_{\mathbf{I}}, g_{\mathbf{I}}(x))$. But $x \in U(f_{\mathbf{I}}, f_{\mathbf{I}}(x))$ and $x \in L(g_{\mathbf{I}}, g_{\mathbf{I}}(x))$ which is a contradiction to (2.8). Hence $f_{\mathbf{I}}(y \rightarrow x) \geq f_{\mathbf{I}}(x)$ and $g_{\mathbf{I}}(y \rightarrow x) \leq g_{\mathbf{I}}(x)$ for all $x, y \in L$. Also, if $f_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) < f_{\mathbf{I}}(x)$ or $g_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) > g_{\mathbf{I}}(x)$ for some $x, y \in L$, then $x \in U(f_{\mathbf{I}}, f_{\mathbf{I}}(x))$ and $x \in L(g_{\mathbf{I}}, g_{\mathbf{I}}(x))$. But $y \rightarrow (x \rightarrow y) \notin U(f_{\mathbf{I}}, f_{\mathbf{I}}(x))$ or $y \rightarrow (x \rightarrow y) \notin L(g_{\mathbf{I}}, g_{\mathbf{I}}(x))$. This is a contradiction to (2.8) and thus $f_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \geq f_{\mathbf{I}}(x)$ and $g_{\mathbf{I}}(y \rightarrow (x \rightarrow y)) \leq g_{\mathbf{I}}(x)$ for all $x, y \in L$. So (3.8) is valid. Consequently, $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ is an IF-ideal of $\mathcal{L} :=(L, \rightarrow, \ell)$ by Theorem 3.2. □

Corollary 3.4. *If* $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ *is an IF-ideal of a unital cycloid* $\mathcal{L} :=(L, \rightarrow, \ell)$*, then the nonempty* $\in_{(t,s)}$*-level set* $\mathbf{I}^{\in}_{(t,s)}$ *is an ideal of* $\mathcal{L} :=(L, \rightarrow, \ell)$ *for all* $(t, s) \in (0, 1] \times [0, 1)$ *satisfying* $t + s \leq 1$*.*

Theorem 3.3 makes it easy to find examples of IF-ideals.

Example 3.5. Let $L = \{\ell, x, \mathfrak{a}\}$ be a set with the Hasse diagram and Cayley table as follows:



$\to$	ℓ	x	$\mathfrak{a}$
ℓ	ℓ	x	$\mathfrak{a}$
x	ℓ	ℓ	$\mathfrak{a}$
$\mathfrak{a}$	ℓ	ℓ	ℓ

Hasse diagram of $(L, \leq_L)$ Cayley table for " $\to$ "

Then $\mathcal{L} :=(L, \to, \ell)$ is a unital cycloid. Define an IF-set $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ in L as follows:

$$f_{\mathbf{I}} : L \to [0,1],\ \ y \mapsto \begin{cases} 0.68 & \text{if } y = \ell, \\ 0.53 & \text{if } y = x, \\ 0.36 & \text{if } y = \mathfrak{a}, \end{cases}$$

and

$$g_{\mathbf{I}} : L \to [0,1],\ \ y \mapsto \begin{cases} 0.27 & \text{if } y = \ell, \\ 0.41 & \text{if } y = x, \\ 0.59 & \text{if } y = \mathfrak{a}. \end{cases}$$

Then the upper t-level set $U(f_{\mathbf{I}}, t)$ and the lower s-level set $L(g_{\mathbf{I}}, s)$ of $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ are calculated as follows:

$$U(f_{\mathbf{I}}, t) = \begin{cases} \varnothing & \text{if } t \in (0.68, 1], \\ \{\ell\} & \text{if } t \in (0.53, 0.68], \\ \{\ell, x\} & \text{if } t \in (0.36, 0.53], \\ L & \text{if } t \in (0, 0.36], \end{cases}$$

and

$$L(g_{\mathbf{I}}, s) = \begin{cases} \varnothing & \text{if } s \in [0, 0.27), \\ \{\ell\} & \text{if } s \in [0.27, 0.41), \\ \{\ell, x\} & \text{if } s \in [0.41, 0.59), \\ L & \text{if } s \in [0.59, 1). \end{cases}$$

It is routine to check that $\{\ell\}$, $\{\ell, x\}$ and L are ideals of $\mathcal{L} :=(L, \to, \ell)$. Therefore $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ is an IF-ideal of $\mathcal{L} :=(L, \to, \ell)$.

Using a collection of ideals, we make an IF-ideal.

Theorem 3.6. *Let $\{I_t \mid t \in \Lambda\}$ be a collection of ideals of a unital cycloid $\mathcal{L} :=(L, \to, \ell)$ such that $L = \bigcup_{t\in\Lambda} I_t$ and*

$$(\forall t, s \in \Lambda)(s > t \ \Leftrightarrow\ I_s \subset I_t),$$

where $\Lambda \subseteq [0,1]$. If we define an IF-set $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ in L by

$$(\forall x \in L) \left(\begin{array}{l} f_{\mathbf{I}}(x) = \sup\{t \in \Lambda \mid x \in I_t\} \\ g_{\mathbf{I}}(x) = \inf\{t \in \Lambda \mid x \in I_t\} \end{array} \right),$$

then $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ is an IF-ideal of $\mathcal{L} :=(L, \to, \ell)$.

Proof. According to Theorem 3.3, it is sufficient to show that $U(f_{\mathbf{I}}, t)$ and $L(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \to, \ell)$ for every $t \in (0, f_{\mathbf{I}}(\ell)]$ and $s \in [g_{\mathbf{I}}(\ell), 1)$. We first consider the two cases:

$$\text{(i) } t = \sup\{r \in \Lambda \mid r < t\}, \text{ (ii) } t \neq \sup\{r \in \Lambda \mid r < t\}.$$

For the first case, we get

$$x \in U(f_{\mathbf{I}}, t) \Leftrightarrow (\forall r < t)(x \in I_r) \Leftrightarrow x \in \bigcap_{r<t} I_r.$$

Then $U(f_{\mathbf{I}}, t) = \bigcap_{r<t} I_r$ which is an ideal of $\mathcal{L} :=(L, \to, \ell)$. For the second case, we want to show $U(f_{\mathbf{I}}, t) = \bigcup_{r\geq t} I_r$. If $x \in \bigcup_{r\geq t} I_r$, then $x \in I_r$ for some $r \geq t$. Thus $f_{\mathbf{I}}(x) \geq r \geq t$. So $x \in U(f_{\mathbf{I}}, t)$. If $x \notin \bigcup_{r\geq t} I_r$, then $x \notin I_r$ for all $r \geq t$. Since $t \neq \sup\{r \in \Lambda \mid r < t\}$, there exists $\varepsilon > 0$ such that $(t - \varepsilon, t) \cap \Lambda = \varnothing$. Thus $x \notin I_r$ for all $r > t - \varepsilon$. So if $x \in I_r$, then $r \leq t - \varepsilon$. Hence $f_{\mathbf{I}}(x) \leq t - \varepsilon < t$, i.e., $x \notin U(f_{\mathbf{I}}, t)$. Therefore $U(f_{\mathbf{I}}, t) = \bigcup_{r\geq t} I_r = I_t$ and it is an ideal of $\mathcal{L} :=(L, \to, \ell)$. In order to prove that $L(g_{\mathbf{I}}, s)$ is an ideal of $\mathcal{L} :=(L, \to, \ell)$, we consider the following cases:

$$\text{(iii) } s = \inf\{r \in \Lambda \mid r > s\}, \text{ (iv) } s \neq \inf\{r \in \Lambda \mid r > s\}.$$

The case (iii) implies

$$x \in L(g_{\mathbf{I}}, s) \Leftrightarrow (\forall r > s)(x \in I_r) \Leftrightarrow x \in \bigcap_{r>s} I_r.$$

So $L(g_{\mathbf{I}}, s) = \bigcap_{r>s} I_r$ which is an ideal of $\mathcal{L} :=(L, \to, \ell)$. The case (iv) implies that there exists $\varepsilon > 0$ such that $(s, s + \varepsilon) \cap \Lambda = \varnothing$. If $x \in \bigcup_{r\leq s} I_r$, then $x \in I_r$ for some $r \leq s$. Thus $g_{\mathbf{I}}(x) \leq r \leq s$. So $x \in L(g_{\mathbf{I}}, s)$. If $x \notin \bigcup_{r\leq s} I_r$, then $x \notin I_r$ for all $r \leq s$ so $x \notin I_r$ for all $r < s + \varepsilon$, that is, if $x \in I_r$, then $r \geq s + \varepsilon$. Thus $g_{\mathbf{I}}(x) \geq s + \varepsilon > s$, i.e., $x \notin L(g_{\mathbf{I}}, s)$. So $L(g_{\mathbf{I}}, s) = \bigcup_{r\leq s} I_r = I_{s_0}$, where $s_0 = \inf\{r \in \Lambda \mid r \leq s\}$, and it is an ideal of $\mathcal{L} :=(L, \to, \ell)$. Consequently, $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ is an IF-ideal of $\mathcal{L} :=(L, \to, \ell)$ by Theorem 3.3. □

Proposition 3.7. *Let* $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ *be an IF-set in a unital cycloid* $\mathcal{L} :=(L, \to, \ell)$. *If* $\mathbf{I}$ *satisfies*

$$\ell_{(t,s)} \in \mathbf{I} \text{ or } \ell_{(t,s)}\, q\, \mathbf{I}, \tag{3.9}$$

$$x_{(t,s)} \in \mathbf{I} \;\Rightarrow \begin{cases} ((x \to y) \to y)_{(t,s)} \in \mathbf{I} \text{ or } ((x \to y) \to y)_{(t,s)}\, q\, \mathbf{I}, \\ (y \to x)_{(t,s)} \in \mathbf{I} \text{ or } (y \to x)_{(t,s)}\, q\, \mathbf{I}, \\ (y \to (x \to y))_{(t,s)} \in \mathbf{I} \text{ or } (y \to (x \to y))_{(t,s)}\, q\, \mathbf{I}, \end{cases} \tag{3.10}$$

for all $x, y \in L$ *and* $(t, s) \in (0, 1] \times [0, 1)$*, then we have*

$$(3.11) \qquad (\forall x \in L)(f_{\mathbf{I}}(\ell) \geq \min\{f_{\mathbf{I}}(x), 0.5\},\ g_{\mathbf{I}}(\ell) \leq \max\{g_{\mathbf{I}}(x), 0.5\}).$$

$$(3.12) \qquad (\forall x, y \in L)\left(\begin{array}{l} f_{\mathbf{I}}((x \to y) \to y) \geq \min\{f_{\mathbf{I}}(x), 0.5\} \\ g_{\mathbf{I}}((x \to y) \to y) \leq \max\{g_{\mathbf{I}}(x), 0.5\} \end{array}\right),$$

$$(3.13) \qquad (\forall x, y \in L)\left(\begin{array}{l} f_{\mathbf{I}}(y \to x) \geq \min\{f_{\mathbf{I}}(x), 0.5\} \\ g_{\mathbf{I}}(y \to x) \leq \max\{g_{\mathbf{I}}(x), 0.5\} \end{array}\right),$$

$$(3.14) \qquad (\forall x, y \in L)\left(\begin{array}{l} f_{\mathbf{I}}(y \to (x \to y)) \geq \min\{f_{\mathbf{I}}(x), 0.5\} \\ g_{\mathbf{I}}(y \to (x \to y)) \leq \max\{g_{\mathbf{I}}(x), 0.5\} \end{array}\right).$$

Also, if $\mathbf{I}$ *satisfies*

$$(3.15) \qquad \begin{array}{l} (\forall x, y \in L)(\forall (t_1, s_1), (t_2, s_2) \in (0, 1] \times [0, 1)) \\ \left(\begin{array}{l} x_{(t_1,s_1)} \in \mathbf{I}, (x \to y)_{(t_2,s_2)} \in \mathbf{I} \ \Rightarrow \\ y_{\min\{(t_1,s_1),(t_2,s_2)\}} \in \mathbf{I} \text{ or } y_{\min\{(t_1,s_1),(t_2,s_2)\}}\, q\, \mathbf{I} \end{array}\right), \end{array}$$

then $f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y), 0.5\}$ *and* $g_{\mathbf{I}}(y) \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y), 0.5\}$ *for all* $x, y \in L$*.*

Proof. Suppose $\mathbf{I}$ satisfies (3.9) and (3.10). If (3.11) is not valid, then

$$f_{\mathbf{I}}(\ell) < t \leq \min\{f_{\mathbf{I}}(x), 0.5\} \text{ or } g_{\mathbf{I}}(\ell) > s \geq \max\{g_{\mathbf{I}}(x), 0.5\}$$

for some $x \in L$ and $(t, s) \in (0, 1] \times [0, 1)$. Thus $(t, s) \in (0, 0.5] \times [0.5, 1)$ and $\ell_{(t,s)} \,\overline{\in}\, \mathbf{I}$. Since $f_{\mathbf{I}}(\ell) + t < 1$ or $g_{\mathbf{I}}(\ell) + s > 1$, we get $\ell_{(t,s)}\, \overline{q}\, \mathbf{I}$. This is a contradiction. So (3.11) is valid. Suppose (3.12) is not true. Then there exist $x, y \in L$ and $(t, s) \in (0, 1] \times [0, 1)$ such that

$$f_{\mathbf{I}}((x \to y) \to y) < t \leq \min\{f_{\mathbf{I}}(x), 0.5\}$$

or $g_{\mathbf{I}}((x \to y) \to y) > s \geq \max\{g_{\mathbf{I}}(x), 0.5\}$. Thus $(t, s) \in (0, 0.5] \times [0.5, 1)$, $((x \to y) \to y)_{(t,s)} \,\overline{\in}\, \mathbf{I}$. Also, $f_{\mathbf{I}}((x \to y) \to y) + t < 2t \leq 1$ since $t \leq 0.5$, or $g_{\mathbf{I}}((x \to y) \to y) + s > 2s \geq 1$ since $s \geq 0.5$. So $((x \to y) \to y)_{(t,s)}\, \overline{q}\, \mathbf{I}$. This is a contradiction. Hence $f_{\mathbf{I}}((x \to y) \to y) \geq \min\{f_{\mathbf{I}}(x), 0.5\}$ and $g_{\mathbf{I}}((x \to y) \to y) \leq \max\{g_{\mathbf{I}}(x), 0.5\}$ for all $x, y \in L$. In the same way as the proof of (3.12), we can prove (3.13) and (3.14). Finally, suppose $\mathbf{I}$ satisfies (3.15). Let $x, y \in L$. Assume that $\min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y)\} < 0.5$ and $\max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y)\} > 0.5$. If $f_{\mathbf{I}}(y) < t < \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y)\}$ or $g_{\mathbf{I}}(y) > s > \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y)\}$, then $(t, s) \in (0, 0.5) \times (0.5, 1)$. It follows that $y_{\min\{(t,s),(t,s)\}} = y_{(t,s)} \,\overline{\in}\, \mathbf{I}$. Also, $y_{\min\{(t,s),(t,s)\}} = y_{(t,s)}\, \overline{q}\, \mathbf{I}$. This is a contradiction. Thus

$$f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y)\} \text{ and } g_{\mathbf{I}}(y) \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y)\}.$$

If $\min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y)\} \geq 0.5$ and $\max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y)\} \leq 0.5$, then

$$x_{\min\{0.5,0.5\}} \in \mathbf{I} \text{ and } (x \to y)_{\min\{0.5,0.5\}} \in \mathbf{I}.$$

It follows from (3.15) that $y_{0.5} = y_{\min\{0.5,0.5\}} \in \mathbf{I}$ or $y_{0.5} = y_{\min\{0.5,0.5\}}\, q\, \mathbf{I}$. So $f_{\mathbf{I}}(y) \geq 0.5$ and $g_{\mathbf{I}}(y) \leq 0.5$. If $f_{\mathbf{I}}(y) < 0.5$ or $g_{\mathbf{I}}(y) > 0.5$, then $f_{\mathbf{I}}(y) + 0.5 < 0.5 + 0.5 = 1$ or $g_{\mathbf{I}}(y) + 0.5 > 0.5 + 0.5 = 1$, Thus $y_{(0.5,0.5)}\, \overline{q}\, \mathbf{I}$. This is a contradiction. So $f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y), 0.5\}$ and $g_{\mathbf{I}}(y) \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y), 0.5\}$ for all $x, y \in L$. □

In a unital cycloid $\mathcal{L} :=(L, \to, \ell)$, if an IF-set $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ satisfies (3.9), (3.10) and (3.15), we say $\mathbf{I}$ an $(\in, \in\vee q)$-*IF-ideal* of $\mathcal{L} :=(L, \to, \ell)$.

Theorem 3.8. *Let $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ be an IF-set in a unital cycloid $\mathcal{L} :=(L, \to, \ell)$. If $\mathbf{I}$ is an $(\in, \in\vee q)$-IF-ideal of $\mathcal{L} :=(L, \to, \ell)$ satisfying $f_{\mathbf{I}}(x) < 0.5 < g_{\mathbf{I}}(x)$ for all $x \in L$, then it is an IF-ideal of $\mathcal{L} :=(L, \to, \ell)$.*

Proof. Let $(t, s) \in (0, 1] \times [0, 1)$ be such that $U(f_{\mathbf{I}}, t) \neq \varnothing \neq L(g_{\mathbf{I}}, s)$. Then there exist $x \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$. Thus $f_{\mathbf{I}}(\ell) \geq \min\{f_{\mathbf{I}}(x), 0.5\} = f_{\mathbf{I}}(x) \geq t$ and $g_{\mathbf{I}}(\ell) \leq \max\{g_{\mathbf{I}}(x), 0.5\} = g_{\mathbf{I}}(x) \leq s$. So $\ell \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$. If $x \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$ and $x \to y \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$, then $f_{\mathbf{I}}(x) \geq t$, $f_{\mathbf{I}}(x \to y) \geq t$, $g_{\mathbf{I}}(x) \leq s$ and $g_{\mathbf{I}}(x \to y) \leq s$. It follows from Proposition 3.7 that $f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y), 0.5\} = \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y)\} \geq t$ and

$$g_{\mathbf{I}}(y) \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y), 0.5\} = \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y)\} \leq s.$$

Thus $y \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$. Let $x \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$. Then $f_{\mathbf{I}}(x) \geq t$ and $g_{\mathbf{I}}(x) \leq s$. Using (3.12), we get $f_{\mathbf{I}}((x \to y) \to y) \geq \min\{f_{\mathbf{I}}(x), 0.5\} = f_{\mathbf{I}}(x) \geq t$ and

$$g_{\mathbf{I}}((x \to y) \to y) \leq \max\{g_{\mathbf{I}}(x), 0.5\} = g_{\mathbf{I}}(x) \leq s.$$

Thus $(x \to y) \to y \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$. By the similar way, we can show that if $x \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$, then $y \to x \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$ and $y \to (x \to y) \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s)$. Thus $U(f_{\mathbf{I}}, t)$ and $L(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \to, \ell)$. So $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ is an IF-ideal of $\mathcal{L} :=(L, \to, \ell)$ by Theorem 3.3. □

Theorem 3.9. *Let $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ be a nonzero-nonunit IF-set in a unital cycloid $\mathcal{L} :=(L, \to, \ell)$, that is, there exists $\mathfrak{a} \in L$ such that $f_{\mathbf{I}}(\mathfrak{a}) \neq 0$ and $g_{\mathbf{I}}(\mathfrak{a}) \neq 1$. If $\mathbf{I}$ is an IF-ideal of $\mathcal{L} :=(L, \to, \ell)$, then the set*

$$\mathbf{I}_0^1 = \{x \in L \mid f_{\mathbf{I}}(x) > 0,\ g_{\mathbf{I}}(x) < 1\}$$

is an ideal of $\mathcal{L} :=(L, \to, \ell)$.

Proof. If $\ell \notin \mathbf{I}_0^1$, then $f_{\mathbf{I}}(\ell) = 0$ or $g_{\mathbf{I}}(\ell) = 1$, so $0 = f_{\mathbf{I}}(\ell) \geq f_{\mathbf{I}}(x)$ or $1 = g_{\mathbf{I}}(\ell) \leq g_{\mathbf{I}}(x)$ for all $x \in L$. Thus $f(x) = 0$ or $g_{\mathbf{I}}(x) = 1$ for all $x \in L$, which is a contradiction. So $\ell \in \mathbf{I}_0^1$. Let $x, y \in L$ be such that $x \in \mathbf{I}_0^1$ and $x \to y \in \mathbf{I}_0^1$. Using Theorem 3.2, we have

$$\begin{aligned}
&f_{\mathbf{I}}(y) \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y)\} > 0,\\
&g_{\mathbf{I}}(y) \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y)\} < 1,\\
&f_{\mathbf{I}}((x \to y) \to y) \geq f_{\mathbf{I}}(x) > 0,\ g_{\mathbf{I}}((x \to y) \to y) \leq g_{\mathbf{I}}(x) < 1,\\
&f_{\mathbf{I}}(y \to x) \geq f_{\mathbf{I}}(x) > 0,\ g_{\mathbf{I}}(y \to x) \leq g_{\mathbf{I}}(x) < 1,\\
&f_{\mathbf{I}}(y \to (x \to y)) \geq f_{\mathbf{I}}(x) > 0,\ g_{\mathbf{I}}(y \to (x \to y)) \leq g_{\mathbf{I}}(x) < 1.
\end{aligned}$$

Hence $y \in \mathbf{I}_0^1$, $(x \to y) \to y \in \mathbf{I}_0^1$, $y \to x \in \mathbf{I}_0^1$ and $y \to (x \to y) \in \mathbf{I}_0^1$. Therefore $\mathbf{I}_0^1$ is an ideal of $\mathcal{L} :=(L, \to, \ell)$. □

Let $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ be an IF-set in a unital cycloid $\mathcal{L} :=(L, \to, \ell)$. For every $(t, s) \in (0, 1] \times [0, 1)$, consider the sets

$$Q(f_{\mathbf{I}}, t) = \{x \in L \mid f_{\mathbf{I}}(x) + t > 1\},\ Q(g_{\mathbf{I}}, s) = \{x \in L \mid g_{\mathbf{I}}(x) + s < 1\}.$$

Theorem 3.10. *Let* $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ *be an IF-set in a unital cycloid* $\mathcal{L} :=(L, \to, \ell)$. *If* $\mathbf{I}$ *is an IF-ideal of* $\mathcal{L} :=(L, \to, \ell)$*, then the nonempty sets* $Q(f_{\mathbf{I}}, t)$ *and* $Q(g_{\mathbf{I}}, s)$ *are ideals of* $\mathcal{L} :=(L, \to, \ell)$ *for all* $(t, s) \in (0, 1] \times [0, 1)$.

Proof. Suppose $Q(f_{\mathbf{I}}, t)$ and $Q(g_{\mathbf{I}}, s)$ are nonempty for all $(t, s) \in (0, 1] \times [0, 1)$. Then there exists $\mathfrak{b} \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. Thus $f_{\mathbf{I}}(\ell) + t \geq f_{\mathbf{I}}(\mathfrak{b}) + t > 1$ and $g_{\mathbf{I}}(\ell) + s \leq g_{\mathbf{I}}(\mathfrak{b}) + s < 1$. So $\ell \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. Let $x, y \in L$. If $x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$ and $x \to y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$, then

$$\begin{aligned}
&f_{\mathbf{I}}(y) + t \geq \min\{f_{\mathbf{I}}(x), f_{\mathbf{I}}(x \to y)\} + t = \min\{f_{\mathbf{I}}(x) + t, f_{\mathbf{I}}(x \to y) + t\} > 1,\\
&g_{\mathbf{I}}(y) + s \leq \max\{g_{\mathbf{I}}(x), g_{\mathbf{I}}(x \to y)\} + s = \max\{g_{\mathbf{I}}(x) + s, g_{\mathbf{I}}(x \to y) + s\} < 1,\\
&f_{\mathbf{I}}((x \to y) \to y) + t \geq f_{\mathbf{I}}(x) + t > 1,\\
&g_{\mathbf{I}}((x \to y) \to y) + s \leq g_{\mathbf{I}}(x) + s < 1,\\
&f_{\mathbf{I}}(y \to x) + t \geq f_{\mathbf{I}}(x) + t > 1,\\
&g_{\mathbf{I}}(y \to x) + s \leq g_{\mathbf{I}}(x) + s < 1,\\
&f_{\mathbf{I}}(y \to (x \to y)) + t \geq f_{\mathbf{I}}(x) + t > 1,\\
&g_{\mathbf{I}}(y \to (x \to y)) + s \leq g_{\mathbf{I}}(x) + s < 1.
\end{aligned}$$

Thus $y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$, $(x \to y) \to y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$, $y \to x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$, and $y \to (x \to y) \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. So $Q(f_{\mathbf{I}}, t)$ and $Q(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \to, \ell)$. □

Proposition 3.11. *Let* $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ *be an IF-set in a unital cycloid* $\mathcal{L} :=(L, \to, \ell)$. *If* $Q(f_{\mathbf{I}}, t)$ *and* $Q(g_{\mathbf{I}}, s)$ *are ideals of* $\mathcal{L} :=(L, \to, \ell)$ *for all* $(t, s) \in (0, 1] \times [0, 1)$*, then the following holds.*

$$(\forall (t, s) \in (0, 0.5] \times [0.5, 1))(\ell \in \mathbf{I}^{\in}_{(t,s)}), \tag{3.16}$$

$$\begin{array}{l}(\forall x, y \in L)(\forall (t_1, s_1), (t_2, s_2) \in (0, 0.5] \times [0.5, 1))\\ (x_{(t_1,s_1)}\, q\, \mathbf{I},\ (x \to y)_{(t_2,s_2)}\, q\, \mathbf{I} \Rightarrow y \in \mathbf{I}^{\in}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}),\end{array} \tag{3.17}$$

$$\begin{array}{l}(\forall x, y \in L)(\forall (t, s) \in (0, 0.5] \times [0.5, 1))\\ \left(x_{(t,s)}\, q\, \mathbf{I} \Rightarrow \left\{ \begin{array}{l} (x \to y) \to y \in \mathbf{I}^{\in}_{(t,s)} \\ y \to x \in \mathbf{I}^{\in}_{(t,s)} \\ y \to (x \to y) \in \mathbf{I}^{\in}_{(t,s)} \end{array} \right. \right).\end{array} \tag{3.18}$$

Proof. Assume that $Q(f_{\mathbf{I}}, t)$ and $Q(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \to, \ell)$. If $\ell \notin \mathbf{I}^{\in}_{(t,s)}$ for some $(t, s) \in (0, 0.5] \times [0.5, 1)$, then $\ell_{(t,s)}\, \overline{\in}\, \mathbf{I}$. Thus $f_{\mathbf{I}}(\ell) < t \leq 1 - t$ or $g_{\mathbf{I}}(\ell) > s \geq 1 - s$. So $\ell \notin Q(f_{\mathbf{I}}, t)$ or $\ell \notin Q(g_{\mathbf{I}}, s)$ which is a contradiction. Hence (3.16) is valid. Let $x, y \in L$ and $(t_1, s_1), (t_2, s_2) \in (0, 0.5] \times [0.5, 1)$ be such that $x_{(t_1,s_1)}\, q\, \mathbf{I}$ and $(x \to y)_{(t_2,s_2)}\, q\, \mathbf{I}$. Then

$$\begin{aligned}
&f_{\mathbf{I}}(x) + \max\{t_1, t_2\} \geq f_{\mathbf{I}}(x) + t_1 > 1,\\
&f_{\mathbf{I}}(x \to y) + \max\{t_1, t_2\} \geq f_{\mathbf{I}}(x \to y) + t_2 > 1,\\
&g_{\mathbf{I}}(x) + \min\{s_1, s_2\} \leq g_{\mathbf{I}}(x) + s_1 < 1,\\
&g_{\mathbf{I}}(x \to y) + \min\{s_1, s_2\} \leq g_{\mathbf{I}}(x \to y) + s_2 < 1,
\end{aligned}$$

which imply that $x \in Q(f_{\mathbf{I}}, \max\{t_1, t_2\})$, $x \to y \in Q(f_{\mathbf{I}}, \max\{t_1, t_2\})$, $x \in Q(g_{\mathbf{I}}, \min\{s_1, s_2\})$ and $x \to y \in Q(g_{\mathbf{I}}, \min\{s_1, s_2\})$. Since $Q(f_{\mathbf{I}}, \max\{t_1, t_2\})$ and $Q(g_{\mathbf{I}}, \min\{s_1, s_2\})$ are ideals of $\mathcal{L} :=(L, \to, \ell)$, it follows that

$$y \in Q(f_{\mathbf{I}}, \max\{t_1, t_2\}) \cap Q(g_{\mathbf{I}}, \min\{s_1, s_2\}).$$

Thus $f_{\mathbf{I}}(y) > 1 - \max\{t_1, t_2\} \geq \max\{t_1, t_2\}$ and

$$g_{\mathbf{I}}(y) < 1 - \min\{s_1, s_2\} \leq \min\{s_1, s_2\}.$$

So $y \in U(f_{\mathbf{I}}, \max\{t_1, t_2\}) \cap L(g_{\mathbf{I}}, \min\{s_1, s_2\}) = \mathbf{I}^{\in}_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}$. Hence (3.17) is valid. Let $x, y \in L$ and $(t, s) \in (0, 0.5] \times [0.5, 1)$ be such that $x_{(t,s)}\, q\, \mathbf{I}$. Then $f_{\mathbf{I}}(x) + t > 1$ and $g_{\mathbf{I}}(x) + s < 1$, so $x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. Since $Q(f_{\mathbf{I}}, t)$ and $Q(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \to, \ell)$, we have $(x \to y) \to y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$, $y \to x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$ and $y \to (x \to y) \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. Thus

$$\begin{aligned} &f_{\mathbf{I}}((x \to y) \to y) > 1 - t \geq t,\ g_{\mathbf{I}}((x \to y) \to y) < 1 - s \leq s,\\ &f_{\mathbf{I}}(y \to x) > 1 - t \geq t,\ g_{\mathbf{I}}(y \to x) < 1 - s \leq s,\\ &f_{\mathbf{I}}(y \to (x \to y)) > 1 - t \geq t,\ g_{\mathbf{I}}(y \to (x \to y)) < 1 - s \leq s, \end{aligned}$$

which imply that $(x \to y) \to y \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s) = \mathbf{I}^{\in}_{(t,s)}$, $y \to x \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s) = \mathbf{I}^{\in}_{(t,s)}$ and $y \to (x \to y) \in U(f_{\mathbf{I}}, t) \cap L(g_{\mathbf{I}}, s) = \mathbf{I}^{\in}_{(t,s)}$. So (3.18) is valid. □

Proposition 3.12. *Let* $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ *be an IF-set in a unital cycloid* $\mathcal{L} :=(L, \to, \ell)$. *If* $Q(f_{\mathbf{I}}, t)$ *and* $Q(g_{\mathbf{I}}, s)$ *are ideals of* $\mathcal{L} :=(L, \to, \ell)$ *for all* $(t, s) \in (0, 1] \times [0, 1)$, *then the following holds.*

$$\begin{aligned} &(\forall x, y \in L)(\forall (t_1, s_1), (t_2, s_2) \in (0.5, 1] \times [0, 0.5))\\ &(x_{(t_1,s_1)} \in \mathbf{I},\ (x \to y)_{(t_2,s_2)} \in \mathbf{I} \ \Rightarrow\ y_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}\, q\, \mathbf{I}), \end{aligned} \tag{3.19}$$

$$\begin{aligned} &(\forall x, y \in L)(\forall (t, s) \in (0.5, 1] \times [0, 0.5))\\ &\left(x_{(t,s)} \in \mathbf{I} \Rightarrow \left\{ \begin{array}{l} ((x \to y) \to y)_{(t,s)}\, q\, \mathbf{I} \\ (y \to x)_{(t,s)}\, q\, \mathbf{I} \\ (y \to (x \to y))_{(t,s)}\, q\, \mathbf{I} \end{array} \right. \right). \end{aligned} \tag{3.20}$$

Proof. Let $x, y \in L$ and $(t_1, s_1), (t_2, s_2) \in (0.5, 1] \times [0, 0.5)$ be such that $x_{(t_1,s_1)} \in \mathbf{I}$ and $(x \to y)_{(t_2,s_2)} \in \mathbf{I}$. Then $f_{\mathbf{I}}(x) \geq t_1 > 1 - t_1$, $f_{\mathbf{I}}(x \to y) \geq t_2 > 1 - t_2$, $g_{\mathbf{I}}(x) \leq s_1 < 1 - s_1$ and $g_{\mathbf{I}}(x \to y) \leq s_2 < 1 - s_2$, which imply that

$$\begin{aligned} &x \in Q(f_{\mathbf{I}}, t_1) \subseteq Q(f_{\mathbf{I}}, \max\{t_1, t_2\}),\ \ x \to y \in Q(f_{\mathbf{I}}, t_2) \subseteq Q(f_{\mathbf{I}}, \max\{t_1, t_2\}),\\ &x \in Q(g_{\mathbf{I}}, s_1) \subseteq Q(g_{\mathbf{I}}, \min\{s_1, s_2\}),\ \ x \to y \in Q(g_{\mathbf{I}}, s_1) \subseteq Q(g_{\mathbf{I}}, \min\{s_1, s_2\}). \end{aligned}$$

Since $Q(f_{\mathbf{I}}, \max\{t_1, t_2\})$ and $Q(g_{\mathbf{I}}, \min\{s_1, s_2\})$ are ideals of $\mathcal{L} :=(L, \to, \ell)$, it follows that $y \in Q(f_{\mathbf{I}}, \max\{t_1, t_2\})$ and $y \in Q(g_{\mathbf{I}}, \min\{s_1, s_2\})$. Thus $f_{\mathbf{I}}(y) + \max\{t_1, t_2\} > 1$ and $g_{\mathbf{I}}(y) + \min\{s_1, s_2\} < 1$. This shows that

$$y_{(\max\{t_1,t_2\},\min\{s_1,s_2\})}\, q\, \mathbf{I}).$$

Now, let $x, y \in L$ and $(t, s) \in (0.5, 1] \times [0, 0.5)$ be such that $x_{(t,s)} \in \mathbf{I}$. Then $f_{\mathbf{I}}(x) \geq t > 1 - t$ and $g_{\mathbf{I}}(x) \leq s < 1 - s$. Thus $x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. Since $Q(f_{\mathbf{I}}, t)$

and $Q(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \to, \ell)$, we have $(x \to y) \to y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$, $y \to x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$, and $y \to (x \to y) \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. So

$$\begin{aligned} &f_{\mathbf{I}}((x \to y) \to y) + t > 1,\ g_{\mathbf{I}}((x \to y) \to y) + s < 1,\\ &f_{\mathbf{I}}(y \to x) + t > 1,\ g_{\mathbf{I}}(y \to x) + s < 1,\\ &f_{\mathbf{I}}(y \to (x \to y)) + t > 1,\ g_{\mathbf{I}}(y \to (x \to y)) + s < 1. \end{aligned}$$

Hence $((x \to y) \to y)_{(t,s)}\, q\, \mathbf{I}$, $(y \to x)_{(t,s)}\, q\, \mathbf{I}$ and $(y \to (x \to y))_{(t,s)}\, q\, \mathbf{I}$. This completes the proof. □

Theorem 3.13. *Let* $\mathbf{I} := \{\langle x, f_{\mathbf{I}}, g_{\mathbf{I}}\rangle \mid x \in L\}$ *be an IF-set in a unital cycloid* $\mathcal{L}$ $:=(L, \to, \ell)$*. If* $\mathbf{I}$ *satisfies*

(3.21) $$(\forall x \in L)(\forall (t,s) \in (0.5,1] \times [0,0.5))(x_{(t,s)}\, q\, \mathbf{I} \Rightarrow \ell_{(t,s)} \in \mathbf{I} \text{ or } \ell_{(t,s)}\, q\, \mathbf{I}),$$

(3.22) $$(\forall x, y \in L)(\forall (t_1,s_1), (t_2,s_2) \in (0.5,1] \times [0,0.5)) \left(\begin{array}{l} x_{(t_1,s_1)}\, q\, \mathbf{I},\ (x \to y)_{(t_2,s_2)}\, q\, \mathbf{I} \Rightarrow \\ y_{(\min\{t_1,t_2\},\max\{s_1,s_2\})} \in \mathbf{I} \text{ or } y_{(\min\{t_1,t_2\},\max\{s_1,s_2\})}\, q\, \mathbf{I}) \end{array} \right),$$

(3.23) $$(\forall x, y \in L)(\forall (t,s) \in (0.5,1] \times [0,0.5)) \left(x_{(t,s)}\, q\, \mathbf{I} \Rightarrow \left\{ \begin{array}{l} ((x \to y) \to y)_{(t,s)} \in \mathbf{I} \text{ or } ((x \to y) \to y)_{(t,s)}\, q\, \mathbf{I}, \\ (y \to x)_{(t,s)} \in \mathbf{I} \text{ or } (y \to x)_{(t,s)}\, q\, \mathbf{I}, \\ (y \to (x \to y))_{(t,s)} \in \mathbf{I} \text{ or } (y \to (x \to y))_{(t,s)}\, q\, \mathbf{I} \end{array} \right. \right),$$

then the nonempty sets $Q(f_{\mathbf{I}}, t)$ *and* $Q(g_{\mathbf{I}}, s)$ *are ideals of* $\mathcal{L} :=(L, \to, \ell)$ *for all* $(t, s) \in (0.5, 1] \times [0, 0.5)$*.*

Proof. Suppose $Q(f_{\mathbf{I}}, t) \neq \varnothing \neq Q(g_{\mathbf{I}}, s)$ for every $(t, s) \in (0.5, 1] \times [0, 0.5)$. If we pick $\mathfrak{c} \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$ such that $f_{\mathbf{I}}(\mathfrak{c}) = g_{\mathbf{I}}(\mathfrak{c}) = 0.5$, then $\mathfrak{c}_{(t,s)}\, q\, \mathbf{I}$. Thus $\ell_{(t,s)} \in \mathbf{I}$ or $\ell_{(t,s)}\, q\, \mathbf{I})$ by (3.21). If $\ell_{(t,s)} \in \mathbf{I}$, then $f_{\mathbf{I}}(\ell) \geq t > 1 - t$ and $g_{\mathbf{I}}(\ell) \leq s < 1 - s$. Thus $\ell \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. It is clear that if $\ell_{(t,s)}\, q\, \mathbf{I}$, then $\ell \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. Let $x, y \in L$ and $(t, s) \in (0.5, 1] \times [0, 0.5)$ be such that $x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$ and $x \to y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. Then $x_{(t,s)}\, q\, \mathbf{I}$ and $(x \to y)_{(t,s)}\, q\, \mathbf{I}$. It follows from (3.22) that $y_{(t,s)} \in \mathbf{I}$ or $y_{(t,s)}\, q\, \mathbf{I}$. It is obvious that if $y_{(t,s)}\, q\, \mathbf{I}$, then $y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. If $y_{(t,s)} \in \mathbf{I}$, then $f_{\mathbf{I}}(y) \geq t > 1 - t$ and $g_{\mathbf{I}}(y) \leq s < 1 - s$, so $y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. Suppose $x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$ and let $y \in L$ and $(t, s) \in (0.5, 1] \times [0, 0.5)$. Then $x_{(t,s)}\, q\, \mathbf{I}$. Using (3.23), we have

(i) $((x \to y) \to y)_{(t,s)} \in \mathbf{I}$ or $((x \to y) \to y)_{(t,s)}\, q\, \mathbf{I}$,
(ii) $(y \to x)_{(t,s)} \in \mathbf{I}$ or $(y \to x)_{(t,s)} \in \mathbf{I}$,
(iii) $(y \to (x \to y))_{(t,s)} \in \mathbf{I}$ or $(y \to (x \to y))_{(t,s)}\, q\, \mathbf{I}$.

For the first case, if $((x \to y) \to y)_{(t,s)}\, q\, \mathbf{I}$, then clearly

$$(x \to y) \to y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s).$$

If $((x \to y) \to y)_{(t,s)} \in \mathbf{I}$, then $f_{\mathbf{I}}((x \to y) \to y) \geq t > 1 - t$ and

$$g_{\mathbf{I}}((x \to y) \to y) \leq s < 1 - s,$$

that is, $(x \to y) \to y \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$. As in the case of (i), for the case of (ii) and (iii), we can show that $y \to x \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s)$ and

$$y \to (x \to y) \in Q(f_{\mathbf{I}}, t) \cap Q(g_{\mathbf{I}}, s).$$

Consequently, $Q(f_{\mathbf{I}}, t)$ and $Q(g_{\mathbf{I}}, s)$ are ideals of $\mathcal{L} :=(L, \to, \ell)$. □

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